J. Fluid Mech. (1971), *vol.* 50, *part* 3, *pp.* 565–577
Printed in Great Britain

Nearly free-molecular slit flow at finite pressure and temperature ratios

By P. Y. WANG† AND E. Y. YU

Department of Mechanical Engineering and Astronautical Science, Northwestern University

(Received 25 January 1971 and in revised form 23 July 1971)

An analytical study is made of nearly free-molecular flow of a noble gas from one reservoir to another through a two-dimensional slit, with finite pressure and temperature ratios across the slit. The fundamental solution of the linear Boltzmann equation is employed in the study. The total mass flow is calculated to the first-order correction terms, of the order of $\alpha \ln \alpha$ and α, where α is the inverse Knudsen number. The coefficients of these terms are in general multiple integrals, but they become explicit functions of the pressure and temperature ratios after the multiple integrations are carried out by using Krook collision model. When the general result is simplified to the isothermal case the first-order correction has a negative value, indicating the reduction of the total mass flow due to intermolecular collisions in the counter flows.

1. Introduction

The steady-state flow of a noble gas from one reservoir to another through an aperture (a circular orifice or a slit) in the separating wall is induced by either a pressure difference or a temperature difference, or a combination of the two, in the equilibrium conditions of the gas at large distances from the aperture. The problem of aperture flow due to a pressure difference only is of long standing beginning with Knudsen's (1909) classical work on 'effusion' in the free-molecular limit. A survey of some of the work on this type of flow will be given later. The problem of aperture flow due to a temperature difference only has not received very much attention although it is connected to the molecular beam work. The more general problem of aperture flow caused by both pressure and temperature differences has apparently not been studied at all. In this general case the net mass flow of the gas could go either way through the aperture and could be zero when the pressure and temperature effects are completely compensated for by each other. In other words, the combined effect of the pressure ratio, $\kappa = p_1/p_2$, and the temperature ratio, $\tau = T_1/T_2$, determines the direction of the net mass flow. Here, $p_1 (= \rho_1 R T_1)$ and T_1 are the equilibrium pressure and temperature of the gas in reservoir 1 and $p_2 (= \rho_2 R T_2)$ and T_2 are those in reservoir 2 (ρ_1 and ρ_2 are the equilibrium densities of the gas in reservoirs 1 and 2 and R is the gas constant). The third parameter in the problem, which determines the flow

† Permanent address: Chung-Shan Institute of Science and Technology, Taipei, Taiwan, The Republic of China.

regime, is the Knudsen number K defined as the ratio of the mean free path of the gas at far upstream in reservoir 1 to the diameter of the orifice or the width of the slit (i.e. $K = \lambda_1/2\tilde{r}_0$, where the tilde denotes a dimensional quantity). These three physical parameters, κ, τ and K, constitute a three-dimensional parameter space encompassing various flow regimes with each regime exhibiting a different flow behaviour. Other parameters may be employed instead of κ and τ. One may define a characteristic velocity U by $U = (p_1/\rho_1)^{\frac{1}{2}}[1-\tau^{\frac{1}{2}}\kappa^{-1}]$ and consequently a Mach number M by $M = U/(p_1/\rho_1)^{\frac{1}{2}} = [1-\tau^{\frac{1}{2}}\kappa^{-1}]$ and a Reynolds number Re by $Re = \rho_1 U\tilde{r}_0/\mu_1 = (\rho_1\tilde{r}_0/\mu_1)(p_1/\rho_1)^{\frac{1}{2}}[1-\tau^{\frac{1}{2}}\kappa^{-1}]$, where μ_1 is the viscosity of the gas far upstream in reservoir 1. Nevertheless, the Mach number and the Reynolds number so defined in the general case are not as meaningful as those corresponding to either the isothermal case ($\tau = 1$) or the isobaric case ($\kappa = 1$). We therefore prefer κ, τ and K to M, Re and $K(= M/Re)$ in this work.

The problem of aperture flow is of interest for several reasons. As pointed out by Liepmann (1961), this problem offers the possibility of a comparison between experiment and a kinetic theory analysis which is not sensitive to the nature of molecular interaction with the boundary walls. Thus it motivates both experimentalist and theoretician to develop methods of obtaining solutions for various regions of the three-dimensional parameter space. Of particular interest are the nearly free-molecular flow region where K is much larger than unity, the slip flow region where K is small and flow regions with either large or small κ and either large or small τ. The method of solution for each region will be different owing to different limiting singular behaviour of flow. Also, the solution of the three-dimensional orifice flow is in general of a different form from that of the two-dimensional slit flow. Furthermore, numerical methods can be developed in computing the mass flow and in studying the effect of various molecular models.

A survey of the literature indicates that most of the experimental, analytical and numerical work is so far centred on the orifice flow at large κ and $\tau = 1$. Liepmann (1961) is the first to do both theoretical and experimental investigations of the mass flow of a noble gas from a large container to a vacuum container through a circular orifice. The containers are of the same temperature and the upstream pressure varies from continuum to free-molecular conditions. Liepmann's experimental data in the nearly free-molecular region agree favourably with the first-order result (including a correction term of the order of K^{-1}) obtained by Narasimha (1961) using Willis's (1958) method. Willis's method is to convert the Krook (Bhatnager, Gross & Krook 1954) kinetic equation into an integral form and to perform iterations on it, starting with the free-molecular value of the distribution function. To improve Narasimha's method of computation, Willis (1965) calculated, by performing one iteration, the total as well as the local mass flow rate of a nearly free-molecular flow through an orifice and through a two-dimensional slit under the same condition of a large pressure ratio across the aperture. There is agreement between Willis's theoretical results and Liepmann's experimental data in the circular orifice case. As far as numerical work is concerned, the method developed by Rotenberg & Weitzner (1969) on hard-sphere molecules is noteworthy. They computed the first-order correction to free-molecular mass flow with an infinite pressure ratio across an orifice.

More experiments on the orifice flow have been performed since Liepmann's work. Sreekanth (1965) undertook his experimental work for the orifice flow at a range of pressure ratios from 1 to 18 and of Knudsen numbers from 0·13 to 1·78. Lord, Hurlbut & Willis (1967) used a different method to repeat Liepmann's experiment, and their result on the total mass flow rate is in good agreement with Willis's orifice solution. Smetana, Sherrill & Schort (1967) made measurements of the discharge characteristic of sharp-edged and round-edged circular orifices for a wide range of Knudsen numbers and pressure ratios. Their results in the case of sharp-edged orifices agree with Liepmann's and Sreekanth's experimental data and with Willis's theoretical results for Knudsen numbers greater than 0·4.

No experiments have yet been performed on the flow through a two-dimensional slit. As mentioned earlier, Willis's (1965) theoretical result on the slit flow is obtained for the limiting case of infinite pressure ratio in which the back flow from reservoir 2 could be ignored. In the case of a finite pressure ratio the only theoretical result which has appeared to date is that of Stewart (1969). By applying Willis's method separately to flows in each direction, Stewart obtained two flow rates under the condition of an infinite pressure ratio and then took the algebraic average of the two values as the rate of the flow with a finite pressure ratio across the slit. Such a procedure is open to question for the simple reason that molecular collisions in the counter flows, which affect the net mass flow, have been completely ignored. It is apparent that the existence of counter flows makes analytical determination of the mass flow difficult. For this reason the problem of aperture flow at finite pressure and temperature ratios has not been treated so far. In this work, however, we shall overcome this difficulty by employing the method of fundamental solution of the linear Boltzmann equation as developed by Yu (1967). Here we will treat nearly free-molecular flow through a two-dimensional slit between two reservoirs at low pressure and temperature ratios. As will be shown later our result is first formally presented in multiple-integral form, valid for a general molecular model. The multiple integrals are then asymptotically expanded in powers of the inverse Knudsen number α (to be defined later along with λ_1). The explicit asymptotic result given here includes only the first-order terms or terms of the order of $\alpha \ln \alpha$ and α and the explicit expression of the coefficients is obtained on the use of the Krook collision model.

2. Fundamental solution

We consider a steady-state flow of a noble gas through a two-dimensional slit of width $\tilde{l} = 2\tilde{r}_0$ between two reservoirs, as shown in figure 1. At a distance far upstream in reservoir 1 the gas is in equilibrium, as characterized by the absolute Maxwellian $F_{m1} = \rho_1 (2\pi R T_1)^{-\frac{3}{2}} \exp(-\xi^2/2RT_1)$, where ξ is the molecular velocity. At some distance far downstream in reservoir 2 the gas reaches another equilibrium state with density ρ_2 and temperature T_2. The wall separating the reservoirs is assumed to be infinitely thin and to be heat insulated in such a way that it has constant temperatures T_1 and T_2 on the sides of reservoirs 1 and 2. The equilibrium pressure in reservoir 1, p_1, is greater than or equal to that in reservoir

2, p_2. The pressure and temperature ratios $\kappa = p_1/p_2$ and $\tau = T_1/T_2$ are assumed to be not much different from unity so that the perturbed distribution function in reservoir 1, $f(= (F - F_{m1})/F_{m1})$, is small compared with unity. The combined effect of these two ratios determines the direction of the net flow, though we consider the net flow from reservoir 1 to 2 to be positive. In the present work, we consider the case of nearly free-molecular flow so that the inverse Knudsen number, the expansion parameter in our problem, is much smaller than unity.

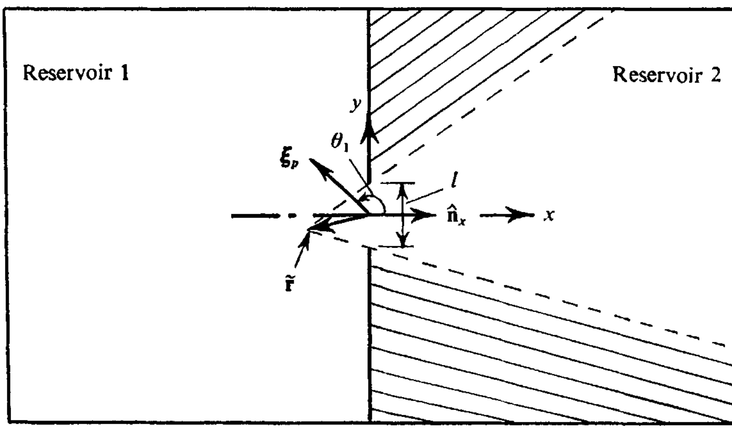

FIGURE 1. Slit flow between two reservoirs, 1 and 2, with ρ_1, T_1, p_1, λ_1, in 1 and ρ_2, T_2, p_2, λ_2 in 2.

Since the slit has a much smaller width than the mean free path of the gas, it appears as a line source with respect to an observer in reservoir 1 at a distance away of the order of λ_1. Molecular particles with a velocity $\boldsymbol{\xi}$ arriving at the observer through the slit are assumed, as an approximation, to be those of the absolute Maxwellian, $F_{m2} = \rho_2(2\pi RT_2)^{-\frac{3}{2}} \exp(-\boldsymbol{\xi}^2/2RT_2)$, from reservoir 2. Molecules arriving there from elsewhere are assumed to be those of F_{m1} under free flow conditions. This is true for any boundary condition at the wall since molecules either diffusely or specularly reflected from the wall have the same Maxwellian distribution function. Using the foregoing assumptions the line source may be written as the product of a two-dimensional delta function $\delta(\tilde{\mathbf{r}})$ [$\tilde{\mathbf{r}}(x = r\cos\theta,\ y = r\sin\theta)$ is the two-dimensional position vector] and a source function. To express the source function with respect to reservoir 1 let us draw a unit vector $\hat{\mathbf{n}}_x$ at the middle point of slit pointing toward 2. Let θ_1 be the angle that $\tilde{\xi}_p$, the planar component of $\boldsymbol{\xi}(= \tilde{\xi}_p\hat{\mathbf{r}} + \boldsymbol{\xi}_z\hat{\mathbf{z}})$, makes with $\hat{\mathbf{n}}_x$ measured counterclockwise from $\hat{\mathbf{n}}_x$ as shown in figure 1. The line source emits molecules towards and absorbs molecules from reservoir 1 at all velocities $\boldsymbol{\xi}$. We consider the molecules emitted by the source toward reservoir 1 (i.e. those of F_{m2} from reservoir 2) to have positive contribution and the molecules from reservoir 1 absorbed by the source (i.e. those flowing through the slit into reservoir 2) to have negative contribution. With this sign convention the source function is written in non-dimensional form (with respect to $RT_1/(\rho_1\lambda_1)$) as

$$\sigma(\boldsymbol{\xi}) = -(2\pi)^{-\frac{3}{2}} 2r_0\xi_p \cos\theta_1 H(-\cos\theta_1)\,[\kappa^{-1}\tau^{\frac{5}{2}} \exp(-\tfrac{1}{2}\tau\xi^2) - \exp(-\tfrac{1}{2}\xi^2)], \quad (1)$$

where $r_0 = \tilde{r}_0/\lambda_1$, $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(RT_1)^{-\frac{1}{2}}$ and $-\cos\theta_1 > 0$. We now add the line source term to the linearized non-dimensionalized Boltzmann equation (Grad 1959) governing the perturbed distribution function $f(\mathbf{r}, \boldsymbol{\xi})$,

$$\boldsymbol{\xi}_p \cdot (\partial f/\partial \mathbf{r}) + \nu(\xi) f = K[f] + (\sigma(\xi)/\omega(\xi))\,\delta(\mathbf{r}), \tag{2}$$

where $\omega(\xi) = (2\pi)^{-\frac{3}{2}} \exp(-\frac{1}{2}\xi^2)$. Equation (2) is written in the form valid for molecules possessing a cut-off potential (Grad 1963). This is because the linear collision integral operator has already been split into two parts with one part involving the collision frequency $\nu(\xi)$ and the other part being the integral

$$K[f] = \int k(\xi, \eta) f(\eta)\, d\eta,$$

where $k(\xi, \eta)$ is a collision kernal. The collision frequency $\nu(\xi)$ has been non-dimensionalized with respect to $(RT_1)^{\frac{1}{2}}/\lambda_1$. Both ν and $k(\xi, \eta)$ have explicit expressions for a given molecular model (Chapman & Cowling 1960; Grad 1963). Now, to solve (2) we decompose f into three parts (Yu 1967) $f = f_\delta + f_a + f_b$, such that each part respectively satisfies the following equations:

$$\boldsymbol{\xi}_p \cdot (\partial f_\delta/\partial \mathbf{r}) + \nu f_\delta = (\sigma(\xi)/\omega(\xi))\,\delta(\mathbf{r}), \tag{3}$$

$$\boldsymbol{\xi}_p \cdot (\partial f_a/\partial \mathbf{r}) + \nu f_a = K[f_\delta], \tag{4}$$

$$\boldsymbol{\xi}_p \cdot (\partial f_b/\partial \mathbf{r}) + \nu f_b = K[f_b] + K[f_a]. \tag{5}$$

Here, f_δ governs those molecules streaming along the ray wedge extending from the line source. Through collisions f_δ produces particles with distribution function f_a. The remaining part f_b governs those particles that have had encounters with particles elsewhere and with the f_a-particles. Integration of (3) along a characteristic yields

$$f_\delta(\mathbf{r}, \boldsymbol{\xi}) = (\sigma(\xi)/\omega(\xi))\,(1/\xi_p) \exp[-\nu(\xi)\, r_\parallel/\xi_p]\, \delta(r_\perp)\, H(r_\parallel), \tag{6}$$

where $r_\perp$ and $r_\parallel$ are the components of $\mathbf{r}$ perpendicular and parallel to $\boldsymbol{\xi}_p$ respectively, and $H(r_\parallel)$ is the Heaviside function. Upon substituting (6) into (4) and then integrating it along a characteristic, we obtain (Yu 1967) f_a in integral form as

$$f_a(r, \theta; \boldsymbol{\xi}) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{\sigma(\eta)}{\omega(\eta)} k(\xi, \eta) \frac{1}{\xi_p} \exp\left\{-\left[\frac{\nu(\xi)\sin\theta'}{\xi_p \sin(\theta' - \theta'')} - \frac{\nu(\eta)\sin\theta''}{\eta_p \sin(\theta' - \theta'')}\right] r\right\}$$
$$\times \frac{1}{|\sin(\theta' - \theta'')|} H\left[\frac{\sin\theta'}{\sin(\theta' - \theta'')}\right] H\left[-\frac{\sin\theta''}{\sin(\theta' - \theta'')}\right] d\theta'\, d\eta_p\, d\eta_z, \tag{7}$$

where θ' and θ'' are the angles that $\boldsymbol{\eta}_p$ and $\boldsymbol{\xi}_p$ make with $\mathbf{r}$.

Finally we solve for the remaining part f_b in an approximate way by expanding it in terms of its moments, $\mathbf{u}_b(r)$ and $p_b(r)$ $(= \rho_b + T_b)$, defined by

$$u_{bx} = \int \xi_x \omega(\xi) f_b\, d\boldsymbol{\xi}, \quad u_{by} = \int \xi_y \omega(\xi) f_b\, d\boldsymbol{\xi}, \quad p_b = \int \tfrac{1}{3}\xi^2 \omega(\xi) f_b\, d\boldsymbol{\xi}, \tag{8}$$

$$\left(\rho_b = \int \omega f_b\, d\boldsymbol{\xi}, \quad T_b = \int \tfrac{1}{3}(\xi^2 - 3)\,\omega f_b\, d\boldsymbol{\xi}\right),$$

where ξ_x and ξ_y are the components of $\boldsymbol{\xi}_p$ parallel and perpendicular to $\hat{\mathbf{n}}_x$ Hence the expansion of f_b takes the form

$$f_b(\mathbf{r}, \boldsymbol{\xi}) = \xi_x u_{bx}(r) + \xi_y u_{by}(\mathbf{r}) + \tfrac{1}{5}\xi^2 p_b(\mathbf{r}). \tag{9}$$

By taking the first moment of (5), integrating the resulting moment equation for p_b with respect to r and using the boundary condition that p_b approaches zero as r approaches infinity we obtain

$$p_b(\mathbf{r}) = -\int \xi_p \cos\theta''' \omega(\xi)\, k(\xi,\zeta)\, k(\zeta,\eta)\, [q p^{-1} \exp(-pr)]\, d\boldsymbol{\eta}\, d\boldsymbol{\zeta}\, d\boldsymbol{\xi}, \qquad (10a)$$

where
$$p = \frac{\nu(\zeta)\sin\theta'}{\zeta_p \sin(\theta'-\theta'')} - \frac{\nu(\eta)\sin\theta''}{\eta_p \sin(\theta'-\theta'')}, \qquad (10b)$$

$$q = \frac{\sigma(\eta)}{\omega(\eta)} \frac{1}{|\sin(\theta'-\theta'')|} H\left[\frac{\sin\theta'}{\sin(\theta'-\theta'')}\right] H\left[\frac{\sin\theta''}{-\sin(\theta'-\theta'')}\right] (\eta_p \zeta_p)^{-1} \qquad (10c)$$

and $\theta' = \cos^{-1}(\hat{\boldsymbol{\eta}}_p . \hat{\mathbf{r}})$, $\theta'' = \cos^{-1}(\hat{\boldsymbol{\zeta}}_p . \hat{\mathbf{r}})$, $\theta''' = \cos^{-1}(\hat{\boldsymbol{\xi}} . \hat{\mathbf{r}})$. (Note that ^ denotes a unit vector.) We then take the zeroth moment of (5) and define $\phi(\mathbf{r})$ as the velocity potential such that $\mathbf{u}_b = -\nabla\phi$. Then ϕ can be shown to satisfy the following Poisson equation

$$\nabla^2\phi = -K^{(0)}.K[q \exp(-pr)], \qquad (11a)$$

where
$$K^{(0)}K[\ldots] = \int \omega(\xi)\, K_{\xi\xi} K_{\zeta\eta} [\ldots]\, d\boldsymbol{\xi} = \int \omega k(\xi,\zeta)\, k(\zeta,\eta)\, [\ldots]\, d\boldsymbol{\eta}\, d\boldsymbol{\zeta}\, d\boldsymbol{\xi}.$$

The boundary condition for (11a) is $\partial\phi/\partial n = 0$, where n denotes the co-ordinate normal to the walls. The Green's function G of (11a) is $G = (2\pi)^{-1}\ln(1/r)$. Hence, the solution of (11a) is given by the following volume integral over the unshaded region V as depicted in figure 1,

$$\phi(\mathbf{r}) = \int_V K^{(0)}.K[q' \exp(-p'r')]\, (2\pi)^{-1} \ln|\mathbf{r}-\mathbf{r}'|^{-1} d\mathbf{r}', \qquad (11b)$$

where p' and q' are the same as (10b, c) except that θ' and θ'' are now measured from $\hat{\mathbf{r}}'$. In (11b) the surface integral has already been dropped as $\partial\phi/\partial n \equiv 0$ and $\partial G/\partial n \equiv 0$ at the boundary. Since V contains both reservoirs care must be exercised in the use of the source function. In the reservoir 1 part of V the source function is that given in (1), whereas in the reservoir 2 part of V the source function used for the computation is the same as that in (1) except that $H(-\cos\theta_1)$ is replaced by $H(\cos\theta_1)$. The components of the velocity $\mathbf{u}_b$ are obtained by differentiating $\phi(\mathbf{r})$ in (11b) with respect to x and y, i.e.

$$\begin{pmatrix} u_{bx} \\ u_{by} \end{pmatrix} = -\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \int_V K^{(0)}.K[q' \exp(-p'r')]\, (2\pi)^{-1} \ln|\mathbf{r}-\mathbf{r}'|^{-1} d\mathbf{r}'. \qquad (11c)$$

In the region near the slit, where r is small, a better approximate solution of f_b is obtained by expanding it in series of the form (Yu 1967)

$$f_b(r,\theta;\boldsymbol{\xi}) = d_0(\boldsymbol{\xi}) + d_1(\theta,\boldsymbol{\xi})\, r \ln r + d_2(\theta,\boldsymbol{\xi})\, r + \ldots. \qquad (12)$$

In the present work we need only evaluate the first term, $d_0(\boldsymbol{\xi}) = f_b(0,\boldsymbol{\xi})$, for a first-order correction to the free-molecular flow rate. Here we approximate $f_b(0,\boldsymbol{\xi})$ by using the moment expansion in (9) evaluated at $r = 0$, i.e.

$$d_0(\boldsymbol{\xi}) = \xi_x u_{bx}(0) + \xi_y u_{by}(0) + \tfrac{1}{6}\xi^2 p_b(0), \qquad (13)$$

where $u_{bx}(0)$, $u_{by}(0)$, and $p_b(0)$ are evaluated as follows. By symmetry of the flow with respect to the centre-line of the slit we have $u_{by}(0) = 0$. From (11 *c*) we find

$$u_{bx}(0) = \lim_{x,\,y\to 0} -\frac{\partial}{\partial x}\int_V K^{(0)}.K[q'\exp[-p'(x'^2+y'^2)^{\frac{1}{2}}]] \times (2\pi)^{-1}\ln[(x-x')^2+(y-y')]^{-\frac{1}{2}}\,dx'\,dy'. \quad (14)$$

Using (10 *a*) we express $p_b(0)$ as

$$p_b(0) = \lim_{r\to 0}\frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} p_b(r,\theta)\,d\theta. \quad (15)$$

3. Evaluation of total mass flow rate

The foregoing results for f_δ in (6), f_a in (7) and f_b in (12) (for small r only) complete the fundamental solution of (2). We will now use the fundamental solution to compute the total mass flow rate through the slit. For this purpose we draw a semicircle of radius $\tilde{r}_0 (\ll \lambda_1)$ with centre at the middle point of the slit, as shown in figures 2 and 3. The radial vector $\tilde{\mathbf{r}}_0$ intercepts the semicircle at a point A and makes an angle θ with the unit normal $\hat{\mathbf{n}}_x$, measured in the

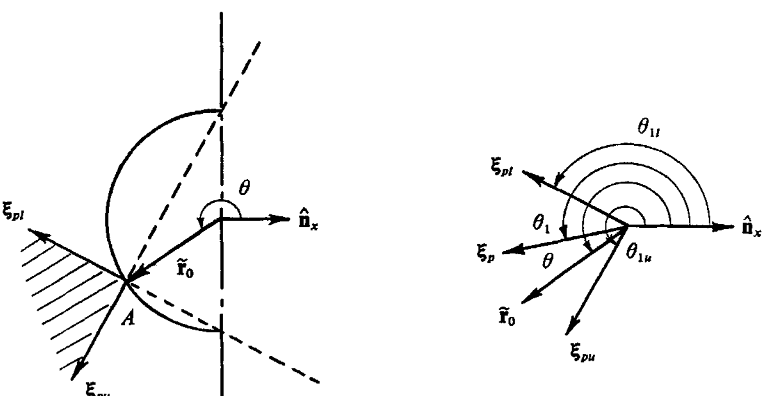

FIGURE 2. Velocity of f_δ-molecules.

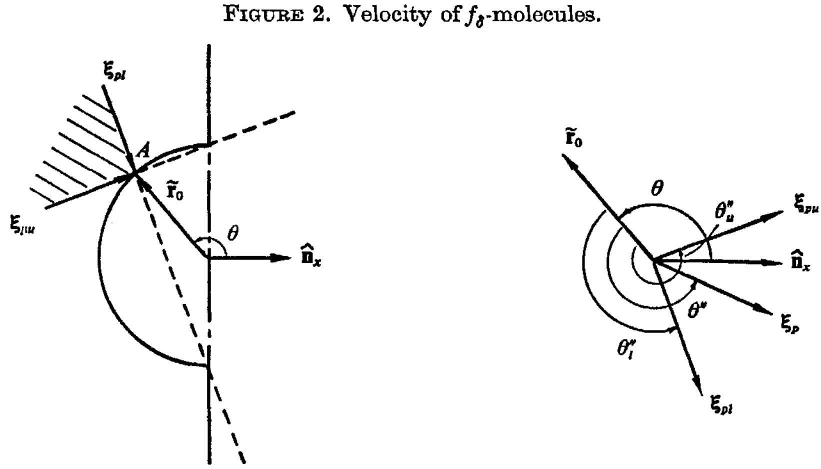

FIGURE 3. Velocity of f_a- and f_b-molecules.

counterclockwise sense. It is easily seen that any molecules reaching the point A on the semicircle must have the direction of their velocities lying between the angular range $[\theta_{1l}, \theta_{1u}]$, where $\theta_{1l} = \frac{1}{4}\pi + \frac{1}{2}\theta$ and $\theta_{1u} = \frac{3}{4}\pi + \frac{1}{2}\theta$ are functions of the position of the point A. The mass flow rate contributed by f_δ, $\dot{m}_\delta$, can be calculated by integrating the first moment of f_δ evaluated at $r = r_0$ over the semicircle:

$$\dot{m}_\delta = -\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int \tilde{\xi}_p \cos(\theta_1 - \theta)\, F_{m1} f_\delta(\mathbf{r}_0, \boldsymbol{\xi})\, d\boldsymbol{\xi}\, \tilde{r}_0\, d\theta. \tag{16}$$

The minus sign in (16) is used to make the flow rate from reservoir 1 to 2 positive. This is necessary since f_δ, as stated earlier, is the distribution function of those molecules directly from the source, at which the direction of velocities towards reservoir 1 was taken to be positive in (1). Upon substituting $\sigma(\xi)$ in (1) and f_δ in (6) into (16) and carrying out the integrations with respect to θ_1 (in the range $(\theta_{1l}, \theta_{1u})$) and θ, we obtain

$$\dot{m}_\delta = (2\pi)^{-\frac{3}{2}} 4\tilde{r}_0 \frac{p_1}{(RT_1)^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_0^{\infty} \{\xi_p^2 [\exp(-\tfrac{1}{2}\xi^2) - \kappa^{-1}\tau^{\frac{5}{2}} \exp(-\tfrac{1}{2}\tau\xi^2)] \times \exp[-\nu(\xi)\,\xi_p^{-1} r_0]\, d\xi_p\, d\xi_z\}. \tag{17a}$$

For small r_0 the double integrations in (17a) may be asymptotically expanded in powers of r_0. The asymptotic expansion will be carried out later, after the molecular model has been specified. At any rate the leading term in the expansion corresponding to $r_0 = 0$ (for $\lambda_1 = \infty$) in the exponent, recovers the free-molecular flow rate, independent of the molecular model, namely,

$$\begin{aligned}\dot{m}_{fm} &= (2\pi)^{-\frac{3}{2}} 4\tilde{r}_0 \frac{p_1}{(RT_1)^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_0^{\infty} \xi_p^2 [\exp(-\tfrac{1}{2}\xi^2) - \kappa^{-1}\tau^{\frac{5}{2}} \exp(-\tfrac{1}{2}\tau\xi^2)]\, d\xi_p\, d\xi_z \\ &= \frac{2\tilde{r}_0}{(2\pi)^{\frac{1}{2}}}\left[\frac{p_1}{(RT_1)^{\frac{1}{2}}} - \frac{p_2}{(RT_2)^{\frac{1}{2}}}\right].\end{aligned} \tag{17b}$$

As will be shown later, the second term in the expansion is of the order of r_0 with the coefficient being a function of τ and κ.

Next we evaluate the mass flow rate corresponding to f_a. As explained earlier f_a governs those molecules in reservoir 1 produced by the f_δ-particles through collisions. It is noted in figure 3 that of the f_a molecules arriving at a point A on the semicircle only the ones with the planar component of the velocity $\boldsymbol{\xi}_p$ lying in the angular range $[\theta_l'' = \frac{5}{4}\pi - \frac{1}{2}\theta, \theta_u'' = \frac{7}{4}\pi - \frac{1}{2}\theta]$ will pass through the slit into reservoir 2. Integration of the first moment of f_a over the entire semicircle yields the mass flow rate $\dot{m}_a$ of those f_a-molecules passing through the slit, namely,

$$\left.\begin{aligned}\dot{m}_a &= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{\theta_l''}^{\theta_u''} \tilde{\xi}_p^2 \cos(\theta'' - \pi)\, F_{m1} f_a\, d\theta''\, d\tilde{\xi}_p\, d\tilde{\xi}_z\, \tilde{r}_0\, d\theta \\ &= \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{\theta_l''}^{\theta_u''}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\Big\{\xi_p \cos\theta'' \frac{p_1}{(RT_1)^{\frac{1}{2}}}\,\omega(\xi)\frac{1}{\omega(\eta)}\,k(\xi,\eta)\,(2\pi)^{-\frac{3}{2}} 2 r_0 \eta_p \\ &\quad \times \cos(\theta + \theta')\, H[-\cos(\theta + \theta')]\,[\kappa^{-1}\tau^{\frac{5}{2}} \exp(-\tfrac{1}{2}\tau\eta^2) - \exp(-\tfrac{1}{2}\eta^2)] \\ &\quad \times \exp\left\{-\left[\frac{\nu(\xi)\sin\theta'}{\xi_p \sin(\theta' - \theta'')} - \frac{\nu(\eta)\sin\theta''}{\eta_p \sin(\theta' - \theta'')}\right] r_0\right\}\frac{1}{|\sin(\theta' - \theta'')|}\, H\left[\frac{\sin\theta'}{\sin(\theta' - \theta'')}\right] \\ &\quad \times H\left[-\frac{\sin\theta''}{\sin(\theta' - \theta'')}\right] d\theta'\, d\eta_p\, d\eta_z\, d\theta''\, d\xi_p\, d\xi_z\, \tilde{r}_0\, d\theta\Big\}.\end{aligned}\right\} \tag{18}$$

The foregoing sevenfold integrations contain the collision frequency $\nu(\xi)$ and the collision kernel $k(\xi, \eta)$, which have not yet been specified. As shown before (Yu 1967), the multiple integrations in (18) can be asymptotically expanded in powers of r_0, for small r_0, with the leading term being of order $r_0 \ln r_0$ and the second term of order r_0^2.

Finally we evaluate the mass flow rate due to f_b as approximated by the expansion (12) evaluated at $r = r_0$,

$$\dot{m}_b = \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{\theta_{4l}}^{\theta_{4u}}\{(RT_1)^{\frac{1}{2}}\xi_p^2 \cos(\theta_4-\pi)\,\omega(\xi)$$
$$\times[\xi_p\cos(\theta_4+\theta)\,u_{bx}(0)+\tfrac{1}{5}\xi^2 p(0)]\,d\theta_4 d\xi_p d\xi_z \tilde{r}_0 d\theta\}+O(r_0^2\ln r_0), \quad (19)$$

where $\theta_4 = \cos^{-1}(\boldsymbol{\xi}_p . \hat{\mathbf{n}}_x)$, $\theta_{4l} = \frac{5}{4}\pi - \frac{1}{2}\theta$, $\theta_{4u} = \frac{7}{4}\pi - \frac{1}{2}\theta$. Note in (19) that only the first term of expansion of f_b, as given in (13), is used while $u_{bx}(0)$ is given in (14) and $p_b(0)$ in (15). By carrying out the integrations in (19) we obtain

$$\dot{m}_b = [p_1 \tilde{l}/2(RT_1)^{\frac{1}{2}}]\,[u_{bx}(0)+(8/5(2\pi)^{\frac{1}{2}})\,p_b(0)]\,r_0+O(r_0^2\ln r_0). \quad (20)$$

4. First-order results

We have now obtained the exact integral expressions of $\dot{m}_\delta$ and $\dot{m}_a$, as given by (17) and (18) respectively, and the series expansion of $\dot{m}_b$,as given by (20). The total mass flow rate per unit width of the slit is the sum, i.e. $\dot{m} = \dot{m}_\delta + \dot{m}_a + \dot{m}_b$. Numerical evaluation of these multiple integrals in the case of a general molecular model appears to be difficult. As we are mainly interested here in obtaining the first-order correction to free-molecular flow rate we shall asymptotically expand the multiple integrals for $\dot{m}_\delta$ and $\dot{m}_a$. However the coefficients of each term of the asymptotic series are still multiple integrals (but of one fold less) and thus cannot in general be evaluated analytically for a general molecular model. Nevertheless, if the Krook molecular model is employed all these multiple integrations can be carried out completely. The coefficients then become explicit functions of the flow parameters κ and τ and thus offer a physical interpretation of the flow problem in terms of these parameters. Furthermore, such an explicit result can be compared with the theoretical solutions of Willis etc. based on the Krook model. The Krook collision kernel in the present case can be shown to be

$$k(\xi, \eta) = \omega(\eta)\,[1+\boldsymbol{\xi}_p . \boldsymbol{\eta}_p + \tfrac{1}{6}(\xi^2-3)(\eta^2-3)]. \quad (21)$$

The corresponding non-dimensional collision frequency ν is a constant which has a value $(8/\pi)^{\frac{1}{2}}$ when λ_1 is defined with respect to the average molecular speed $\bar{\xi} = (8/\pi)^{\frac{1}{2}}(RT_1)^{\frac{1}{2}}$. For convenience we define the inverse Knudsen number as $\alpha = \nu r_0 = (8/\pi)^{\frac{1}{2}} r_0 = (2/\pi)^{\frac{1}{2}} K^{-1}$.

Now, by using the Krook molecular model, we expand the integrals in (17a) and (18) and evaluate the coefficients analytically, obtaining

$$\dot{m}_\delta = \dot{m}_{fm} - \frac{2\tilde{r}_0}{\pi}\left[\frac{p_1}{(RT_1)^{\frac{1}{2}}} - \frac{p_2}{(RT_2)^{\frac{1}{2}}}\left(\frac{T_1}{T_2}\right)^{\frac{1}{2}}\right]\alpha + O(\alpha^2). \quad (22)$$

$$\dot{m}_a = [p_1 2\tilde{r}_0/4\pi(RT_1)^{\frac{1}{2}}]$$
$$\times\{B(\kappa,\tau)\,\alpha\ln\alpha^{-1} - [\kappa^{-1}I(\tau)-I(1)+0{\cdot}577\,B(\kappa,\tau)]\,\alpha + O(\alpha^2\ln\alpha)\}, \quad (23)$$

where
$$B(\kappa,\tau) = \tfrac{1}{2}\pi - 1 - \kappa^{-1}(\tfrac{1}{2}\pi\tau^{\frac{1}{2}} - \tau) \tag{24}$$
and
$$I(\tau) = (2-1/3\tau)\,\tau[2U_{11}(\tau)+0\cdot442] - \tau^{\frac{1}{2}}[8U_{22}+0\cdot090] - \tfrac{1}{3}[8U_{31}+0\cdot384] - (\tfrac{1}{2}-1/6\tau)\,\tau[8U_{13}+0\cdot384] + \tfrac{1}{6}[48U_{33}-0\cdot567], \tag{25}$$
with
$$U_{mn}(\tau) = \int_0^{\frac{1}{2}\pi} \cos^m\theta\sin^n\theta\ln(\cos\theta+\tau^{\frac{1}{2}}\sin\theta)\,d\theta, \tag{26}$$

which can be numerically evaluated on a computer for a given value of τ. At $\tau = 1$ we find
$$I(\tau) = 0\cdot1225. \tag{27}$$

In a similar way we carry out the 11-fold integrations for $u_{bx}(0)$ in (14) and the 10-fold integrations for $p_b(0)$ in (15) and substitute the results into (20), thus obtaining
$$\dot{m}_b = (p_1(2\tilde{r}_0)/(2\pi RT_1)^{\frac{1}{2}})\{(\tfrac{1}{2}\pi)^{\frac{1}{2}}[R(1)-\kappa^{-1}R(\tau)] + \tfrac{4}{5}[P(1)-\kappa^{-1}P(\tau)]\}\,\alpha + O(\alpha^2\ln\alpha). \tag{28a}$$

Here, $$R(\tau) = -2\pi^{-3}[(4\tau-\tfrac{2}{3})\,R_{21}(\tau) - 8\tau^{\frac{1}{2}}R_{32} - \tfrac{8}{3}R_{41} - 4(\tau-\tfrac{1}{3})\,R_{23} + 8R_{43}], \tag{28b}$$
$$P(\tau) = -(2/\pi^7)^{\frac{1}{2}}[\tfrac{1}{3}(7\tau-1)\,R_{21}(\tau) + \tfrac{3}{8}\pi^2(2\tau-\tfrac{1}{3})\,R_{22} - \tfrac{16}{3}\tau^{\frac{1}{2}}R_{32} - \tfrac{4}{3}R_{41} + \tfrac{4}{9}R_{23} - \tfrac{15}{8}\pi^2\tau^{\frac{1}{2}}R_{33} - \tfrac{5}{8}\pi^2R_{42} - \tfrac{15}{16}\pi^2(\tau-\tfrac{1}{3})\,R_{24} + \tfrac{8}{3}R_{43} - 8\tau^{\frac{1}{2}}R_{34} - 4(\tau-\tfrac{1}{3})\,R_{25} + \tfrac{35}{16}\pi^2R_{44} + \tfrac{32}{3}R_{45}, \tag{28c}$$
where
$$R_{mn}(\tau) = \int_0^{\frac{1}{2}\pi} \frac{\cos^m\theta\sin^n\theta}{\cos\theta+\tau^{\frac{1}{2}}\sin\theta}\,d\theta$$

and can be evaluated numerically for a given value of τ. At $\tau = 1$, $R(1) = 0\cdot0184$ and $P(1) = 0\cdot0224$.

Finally, by adding $\dot{m}_s$ in (22), $\dot{m}_a$ in (23) and $\dot{m}_b$ in (28), and taking into account terms of the order of $\alpha\ln\alpha^{-1}$ and α we obtain the total mass flow rate per unit width of the slit,
$$\dot{m} = \dot{m}_{fm} + \dot{m}_{fm}^{\infty}[(8\pi)^{-\frac{1}{2}}B(\kappa,\tau)]\,\alpha\ln\alpha^{-1} - \dot{m}_{fm}^{\infty}\{(2/\pi)^{\frac{1}{2}}(1-\kappa^{-1}\tau) + [0\cdot577(8\pi)^{-\frac{1}{2}}]\,B(\kappa,\tau) - (8\pi)^{-\frac{1}{2}}[I(1)-\kappa^{-1}I(\tau)] - (\tfrac{1}{2}\pi)^{\frac{1}{2}}[R(1)-\kappa^{-1}R(\tau)] - \tfrac{4}{5}[P(1)-\kappa^{-1}P(\tau)]\}\,\alpha + O(\alpha^2\ln\alpha,\alpha^2), \tag{29}$$
where $\dot{m}_{fm} = [\tilde{l}p_1/(2\pi RT_1)^{\frac{1}{2}}]\,[1-\tau^{\frac{1}{2}}K^{-1}]$ is that given in (17b) and
$$\dot{m}_{fm}^{\infty} = \tilde{l}p_1/(2\pi RT_1)^{\frac{1}{2}}$$
is the rate of flow into a vacuum reservoir (with $p_2 = 0$ or $K = \infty$). The subscripts fm signify that $\dot{m}$ and $\dot{m}^{\infty}$ have the same expressions as those in the free-molecular flow condition. It should be noted that in the isothermal case the pressures under the free flow condition are lower than the values here for the same slit width. Now, when $p_1 = p_2$ or $\kappa = 1$ and $T_1 = T_2$ or $\tau = 1$, $\dot{m}$ in (29) vanishes identically, as it should. When $T_1 = T_2$ in the isothermal flow $\dot{m}_{fm}|_{\tau=1} = (1-\kappa^{-1})\,\dot{m}_{fm}^{\infty}$ and $\dot{m}$ in (29) reduces to
$$\dot{m}|_{\tau=1} = \dot{m}_{fm}|_{\tau=1}\{1 + (1/2(2\pi)^{\frac{1}{2}})\,(\tfrac{1}{2}\pi-1)\,\alpha\ln\alpha^{-1} - 0\cdot7975\,\alpha + O(\alpha^2\ln\alpha,\alpha^2)\}. \tag{30}$$

It is noted from (30) that $\dot{m}/\dot{m}_{fm}$, in the case of $\tau = 1$, is independent of κ, as shown by a single curve plotted in figure 4. This plot indicates that $(\dot{m}-\dot{m}_{fm})/\dot{m}_{fm}$ is negative.

When $p_1 = p_2$ in the isobaric case, the flow is induced by thermal diffusion only. In this case $\dot{m}_{fm}|_{\kappa=1} = [\tilde{l}p_1/(2\pi RT_1)^{\frac{1}{2}}][1-\tau^{\frac{1}{2}}]$, where the factor

$$\tilde{l}p_1/(2\pi RT_1)^{\frac{1}{2}}$$

no longer has the same meaning as in the case of an infinite pressure ratio. After a simplication of (29) the total mass flow in the isobaric case is given by

$$\begin{aligned}\dot{m}|_{\kappa=1} = \dot{m}_{fm}|_{\kappa=1}\{&1+(8\pi)^{-\frac{1}{2}}[\tfrac{1}{2}\pi-(1-\tau)/(1-\tau^{\frac{1}{2}})]\,\alpha\ln\alpha^{-1}-(1-\tau^{\frac{1}{2}})^{-1}\\ &\times\{(2/\pi)^{\frac{1}{2}}(1-\tau)+0{\cdot}577(8\pi)^{-\frac{1}{2}}[\tfrac{1}{2}\pi(1-\tau^{\frac{1}{2}})-(1-\tau)]-(8\pi)^{-\frac{1}{2}}[I(1)-I(\tau)]\\ &-(\tfrac{1}{2}\pi)^{\frac{1}{2}}[R(1)-R(\tau)]-\tfrac{4}{5}[P(1)-P(\tau)]\}\alpha+O(\alpha^2\ln\alpha,\alpha^2). \end{aligned} \quad (31)$$

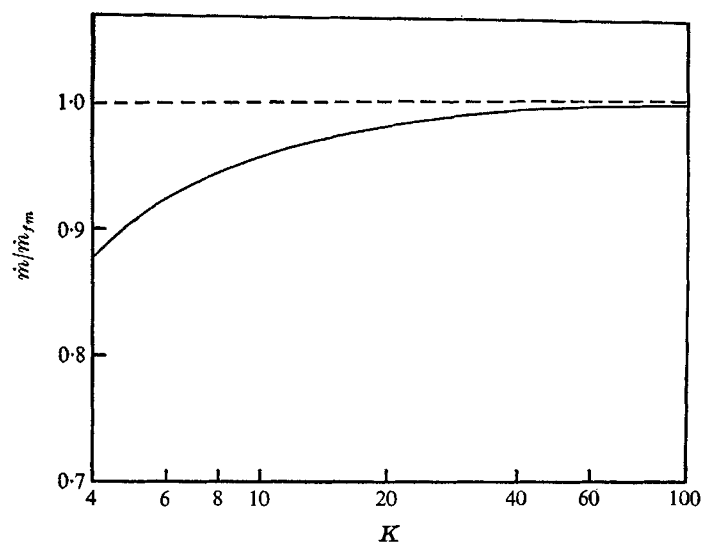

FIGURE 4. $[\dot{m}/\dot{m}_{fm}]_{\tau=1}$ *vs.* K.

The sign of $\dot{m}|_{\kappa=1}$ follows that of $\dot{m}_{fm}|_{\kappa=1}$ at a given value of τ. When $\tau < 1$ or $T_1 < T_2$ (or $\rho_1 > \rho_2$) both $\dot{m}$ and $\dot{m}_{fm}$ are positive, indicating a net flow from reservoir 1 to 2. When $\tau > 1$ or $T_1 > T_2$ (or $\rho_1 < \rho_2$), the direction of the net flow is reversed. A plot of $[\dot{m}|\dot{m}_{fm}]_{\kappa=1}\,(>0)$ versus K is made in figure 5 for various values of τ in the range 0·5 to 2·0, and shows that the net flow rate decreases from the free flow value as the Knudsen number K decreases or as more molecular collisions take place. The effect of τ on the direction of flow, as already stated, can be seen from the relation

$$\dot{m}|_{\kappa=1}/[\tilde{l}p_1/(2\pi RT_1)^{\frac{1}{2}}] = (1-\tau^{\frac{1}{2}})\,[\dot{m}/\dot{m}_{fm}]_{\kappa=1}.$$

To see the overall effect of κ and τ on the total mass flow rate it is necessary to take the general result given by (29). The flow could go either way, depending on the values of these two parameters. To this end it should be pointed out that the α term in our asymptotic solution is of the same order of magnitude as the $\alpha\ln\alpha$ term and thus cannot be neglected.

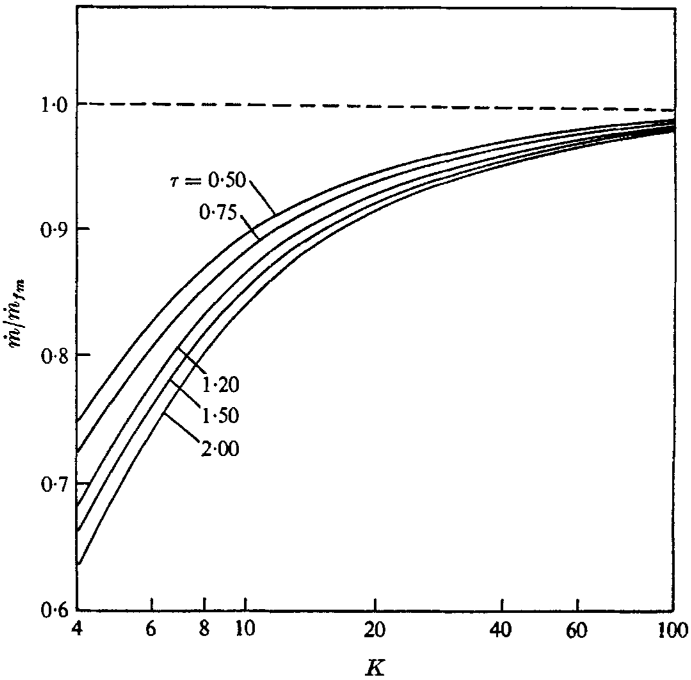

FIGURE 5. $[\dot{m}/\dot{m}_{fm}]_{K=1}$ *vs.* K, $\dot{m} > \dot{m}_{fm}$, $\dot{m} = \dot{m}_{fm}$, $\dot{m} < \dot{m}_{fm}$ for $\tau < 1$, $\tau = 1$, $\tau > 1$ respectively.

5. Conclusions and discussions

It is of interest to compare our result, given in (30), for the case of $\tau = 1$ with that obtained by Willis (1965). Willis's result is written in terms of our α as

$$\dot{m}^{\infty}|_{\tau=1} = \dot{m}^{\infty}_{fm}|_{\tau=1}\{1 + (1/2(2\pi)^{\frac{1}{2}})\,(\tfrac{1}{2}\pi - 1)\,\alpha[\ln \alpha^{-1} + \tfrac{1}{2}\ln 2]\}. \quad (32)$$

The coefficient of the $\alpha \ln \alpha$ term in (32) for $[\dot{m}^{\infty}/\dot{m}_{fm}]_{\tau=1}$ is exactly the same as that in (30). This coincidence is rather astonishing in view of the fact that our solution is based on the linearized Krook equation for a finite pressure ratio whereas Willis' result is obtained by iterating from the integral form of the non-linear Krook equation for an infinite pressure ratio. This indicates that the percentage correction, $(\dot{m} - \dot{m}_{fm})/\dot{m}_{fm}$ or $(\dot{m}^{\infty} - \dot{m}^{\infty}_{fm})/\dot{m}^{\infty}_{fm}$, is the same up to the $\alpha \ln \alpha$ term, irrespective of the pressure ratio across the slit. Our result, given by (30), indicates further that the percentage correction is independent of the pressure ratio even up to the α term. The difference between our first-order result in (30) and Willis's result in (32) is that our result gives a net negative correction in the finite pressure ratio case, whereas Willis's result gives a net positive correction in the infinite pressure case. A negative correction appears to be reasonable since intermolecular collisions in the counter flows between the two reservoirs should reduce the flow rate from the free-molecular value.

In conclusion, it is suggested that experiments on an aperture (slit or orifice) flow in the finite pressure ratio case be conducted so that a meaningful comparison can be made between measurements and our theoretical result here.

This work was supported by the National Science Foundation under Grant No. GK-2831. Financial assistance to one of the authors (P.Y.W.) in the form of a fellowship from Chung-Shan Institute of Science and Technology, Taipei, Taiwan, The Republic of China, is sincerely appreciated.

REFERENCES

BHATNAGER, P. L., GROSS, E. P. & KROOK, M. 1954 *Phys. Rev.* **94**, 511.

CHAPMAN, S. & COWLING, T. G. 1960 *The Mathematical Theory of Non-Uniform Gases*, chap. 7. Cambridge University Press.

GRAD, H. 1959 In *Proc. Symp. on Aerodynamics of Upper Atmosphere*, Project Rand.

GRAD, H. 1963 In *Rarefied Gas Dynamics*, vol. 1 (ed. J. A. Laurmann) p. 26. Academic.

KNUDSEN, M. 1909 *Ann. Phys.* **28**, 75.

LIEPMANN, H. W. 1961 *J. Fluid Mech.* **10**, 65.

LORD, R. G., HURLBUT, F. C. & WILLIS, D. R. 1967 In *Rarefied Gas Dynamics*, vol. 2 (ed. C. L. Brundin) p. 1235. Academic.

NARASIMHA, R. 1961 *J. Fluid Mech.* **10**, 371.

ROTENBERG, A. & WEITZNER, H. 1969 *Phys. Fluids*, **12**, 1573.

SMETANA, F. O., SHERRILL, W. A. & SCHORT, D. R. 1967 In *Rarefied Gas Dynamics*, vol. 2 (ed. C. L. Brundin) p. 1243. Academic.

SREEKANTH, A. K. 1965 In *Rarefied Gas Dynamics*, vol. 1 (ed. J. H. de Leeuw) p. 621. Academic.

STEWART, J. D. 1969 *J. Fluid Mech.* **35**, 599.

WILLIS, D. R. 1958 *Princeton University Aeronaut. Eng. Rep.* no. 442.

WILLIS, D. R. 1965 *J. Fluid Mech.* **21**, 21.

YU, E. Y. 1967 *Phys. Fluids*, **10**, 2466.